THE PAPUA NEW GUINEA UNIVERSITY OF TECHNOLOGY

DEPARTMENT OF MATHEMATICS & COMPUTER SCIENCE

FIRST SEMESTER EXAMINATIONS - 2023

FIRST YEAR BACHELOR OF APPLIED MATHEMATICS

AM112 – CALCULUS AND ALGEBRA

TIME ALLOWED: 3 HOURS

INFORMATION FOR CANDIDATES

1. Write your name and student number clearly on the front of the examination booklet.
2. You have 10 minutes to read this paper. You must not begin writing during this time.
3. **Answer Question 6 (it is compulsory), and any other four (4) questions**.
4. All answers must be written in examination booklets only. No other written material will be accepted.
5. Start the answer for each question on a **new** page. Do **not** use red ink.
6. Notes and textbooks are not allowed in the examination room. All mobile phones and electronic/recording devices must be switched off during the examination.
7. Scientific calculators are allowed in the examination room.
8. A formula sheet is attached.

MARKING SCHEME

Marks are indicated at the beginning of each question. All questions carry equal marks.

Question 1 **FUNCTIONS AND LIMITS (10 marks)**

a) Given the identity $(\log_b a)(\log_a b) = 1$. Solve $(\log_2 81)(\log_3 32)$ without using a calculator. Show your working out. **(2 marks)**

b) What is the amplitude of $f(x) = \frac{1}{2}\cos x$? **(2 marks)**

c) If $\tanh x_0 = \frac{4}{5}$ what is $\sinh x_0$, and $\cosh x_0$? **(2 marks)**

d) Find $\lim_{x\to 2} \frac{x^2-4}{x-2}$ **(2 marks)**

e) First rationalize the numerator, then find $\lim_{x\to 0} \frac{\sqrt{x^2+4}-2}{x}$ **(2 marks)**

Question 2 **DERIVATIVES (10 marks)**

a) Find $\frac{dy}{dx}$ of the implicit function, $\sin(x^2y^2) = x$ (Hint: You can use method of substitution combined with the product rule to solve). **(5 marks)**

b) Find relative extrema using both the first and second derivative test, of $f(x) = \frac{1}{2}x - \sin x,\ \ 0 < x < 2\pi$. **(5 marks)**

Question 3 **INTEGRATION (10 marks)**

a) Solve $I = \int \left(3x^2 + 1 + \frac{1}{x^2+x-2}\right) dx$. Complete integration of the 3rd term in the integral by partial fractions and complete the whole problem. **(3 marks)**

b) solve: $\int \sec^7 x \tan^3 x\, dx$. **(3 marks)**

c) Find the arc length of the curve $y = \frac{x^2}{2}$ from $x = 0$ to $x = 1$. Arc length of a curve is given by the formula $L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$. Solve using trigonometric substitution. Draw the triangle to indicate the relevant working out. DO NOT USE ANY OTHER METHOD. Helpful table is given. **(4 marks)**

Question 4 **COMPLEX NUMBERS (10 marks)**

a) Find and **plot all roots** of the following complex number $\sqrt[3]{2+2i}$. **(7 marks)**

b) Express each root in its rectangular, polar and exponential form. **(3 marks)**

Question 5 **MATRICES (10 marks)**

a) Calculate the inverse from $\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}[A_{jk}]^T$ where A_{jk} is the minor of a_{jk} in $\det \mathbf{A}$ (adjoint) (3 marks). Check by using $\mathbf{AA}^{-1} = \mathbf{I}$, (Show all steps) (2 marks).

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & -1 & 2 \\ 2 & 4 & 11 \end{bmatrix}$$

(5 marks in total)

(b) Find the eigenvalues and eigenvectors of the following matrix. You should obtain two eiganvectors. Show that you can get four eiganvectors (4 marks) but two pairs are equivalent (1 mark).

$$\begin{bmatrix} 5 & -2 \\ 9 & 6 \end{bmatrix}$$

(5 marks in total)

Question 6 **VECTORS (10 marks)**

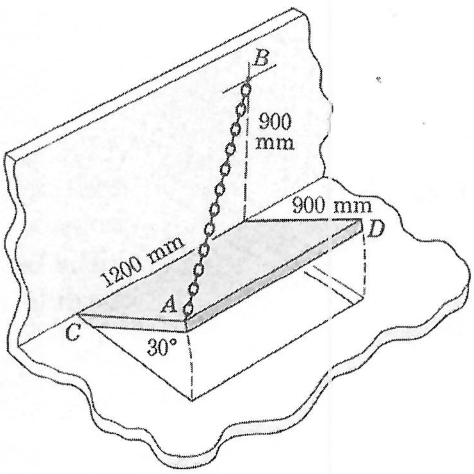
a) Find the work done by a force $\mathbf{p} = [2, 6, 6]$ acting on a body if the body is displaced from a point to a point B along the straight segment AB, where A: $(3, 4, 0)$ & B: $(5, 8, 0)$. Sketch $\mathbf{p}$ and AB. Show the details of your work. **(2 marks)**

b) In the following w.r.t a right-handed Cartesian coordinate system, let $\mathbf{a} = [1, 2, 0]$, $\mathbf{b} = [-3, 2, 0]$. Find the following expressions.

i) $\mathbf{a} \times \mathbf{b}$, **(2 marks)**

ii) $\mathbf{a} \cdot \mathbf{b}$ **(1 marks)**

c)



The access door is held in the 30° open position by the chain AB. If the tension in the chain is 100 N, determine the projection of the tension force on the diagonal axis CD of the door.

$(Hint: F_{CD} = \overrightarrow{F_{AB}} \cdot \overrightarrow{n_{CD}})$

(5 marks)

DATA SHEET for AM112 EXAMS 2023 SEMESTER 1

Trigonometrical identities

(a) $\sin^2\theta + \cos^2\theta = 1$; $\sec^2\theta = 1 + \tan^2\theta$; $\csc^2\theta = 1 + \cot^2\theta$

(b) $\sin(A+B) = \sin A\cos B + \cos A\sin B$

$\sin(A-B) = \sin A\cos B - \cos A\sin B$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\cos(A-B) = \cos A\cos B + \sin A\sin B$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

(c) Let $A = B = \theta$ $\therefore$ $\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

Hyperbolic identities

$\cosh x + \sinh x = e^x$	$\sinh(x+y) = \sinh x\cosh y + \cosh x\sinh y$
$\cosh x - \sinh x = e^{-x}$	$\cosh(x+y) = \cosh x\cosh y + \sinh x\sinh y$
$\cosh^2 x - \sinh^2 x = 1$	$\sinh(x-y) = \sinh x\cosh y - \cosh x\sinh y$
$1 - \tanh^2 x = \operatorname{sech}^2 x$	$\cosh(x-y) = \cosh x\cosh y - \sinh x\sinh y$
$\coth^2 x - 1 = \operatorname{csch}^2 x$	$\sinh 2x = 2\sinh x\cosh x$
$\cosh(-x) = \cosh x$	$\cosh 2x = \cosh^2 x + \sinh^2 x$
$\sinh(-x) = -\sinh x$	$\cosh 2x = 2\sinh^2 x + 1 = 2\cosh^2 x - 1$

Derivatives and Integrals

1 $\frac{d}{dx}(x^n) = nx^{n-1}$ $\quad\therefore \int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ $\left\{\begin{matrix}\text{provided}\\ n \neq -1\end{matrix}\right\}$

2 $\frac{d}{dx}(\ln x) = \frac{1}{x}$ $\quad\therefore \int \frac{1}{x}\,dx = \ln x + C$

3 $\frac{d}{dx}(e^x) = e^x$ $\quad\therefore \int e^x\,dx = e^x + C$

4 $\frac{d}{dx}(e^{kx}) = ke^{kx}$ $\quad\therefore \int e^{kx}\,dx = \frac{e^{kx}}{k} + C$

5 $\frac{d}{dx}(a^x) = a^x\ln a$ $\quad\therefore \int a^x\,dx = \frac{a^x}{\ln a} + C$

6 $\frac{d}{dx}(\cos x) = -\sin x$ $\quad\therefore \int \sin x\,dx = -\cos x + C$

7 $\frac{d}{dx}(\sin x) = \cos x$ $\quad\therefore \int \cos x\,dx = \sin x + C$

8 $\frac{d}{dx}(\tan x) = \sec^2 x$ $\quad\therefore \int \sec^2 x\,dx = \tan x + C$

More derivatives

$\frac{d}{dx}[\tan x] = \sec^2 x$ $\qquad \frac{d}{dx}[\sec x] = \sec x\tan x$

$\frac{d}{dx}[\cot x] = -\csc^2 x$ $\qquad \frac{d}{dx}[\csc x] = -\csc x\cot x$

Specific integrals

$$\int \tan x\,dx = \ln|\sec x| + C$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$

TRIGONOMETRIC SUBSTITUTIONS

EXPRESSION IN THE INTEGRAND	SUBSTITUTION	RESTRICTION ON θ	SIMPLIFICATION
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$-\pi/2 \le \theta \le \pi/2$	$a^2 - x^2 = a^2 - a^2\sin^2\theta = a^2\cos^2\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$-\pi/2 < \theta < \pi/2$	$a^2 + x^2 = a^2 + a^2\tan^2\theta = a^2\sec^2\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$\begin{cases}0 \le \theta < \pi/2 & (\text{if } x \ge a)\\ \pi/2 < \theta \le \pi & (\text{if } x \le -a)\end{cases}$	$x^2 - a^2 = a^2\sec^2\theta - a^2 = a^2\tan^2\theta$

Reduction formula

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx$$

$$\int \tan^n x\,dx = \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x\,dx$$

$$\int \sec^n x\,dx = \frac{\sec^{n-2}x\tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx$$

- Equation of a straight line: $y - y_1 = m(x - x_1)$
- The gradient of the normal is equivalent to the negative reciprocal of the gradient of a tangent.
- Radius of curvature, $R = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$
- Position of centre of curvature (h, k) $h = x_1 - R\sin\theta$, $k = y_1 + R\cos\theta$

INTEGRATING PRODUCTS OF TANGENTS AND SECANTS

$\int \tan^m x \sec^n x\, dx$	PROCEDURE	RELEVANT IDENTITIES
n even	• Split off a factor of $\sec^2 x$. • Apply the relevant identity. • Make the substitution $u = \tan x$.	$\sec^2 x = \tan^2 x + 1$
m odd	• Split off a factor of $\sec x \tan x$. • Apply the relevant identity. • Make the substitution $u = \sec x$.	$\tan^2 x = \sec^2 x - 1$
$\begin{cases} m \text{ even} \\ n \text{ odd} \end{cases}$	• Use the relevant identities to reduce the integrand to powers of $\sec x$ alone. • Then use the reduction formula for powers of $\sec x$.	$\tan^2 x = \sec^2 x - 1$

- Volume by cylindrical shells: $V_y = \int_a^b 2\pi x f(x) dx$ or $V_x = \int_a^b 2\pi y f(y) dy$
- Length of a curve $L = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$
- Eigenvalues and eigenvectors:
  $\mathbf{Ax} = \lambda\mathbf{x}$ & $(\mathbf{A} - \lambda\mathbf{I}) = \mathbf{0}$
- In vector algebra: $\vec{\boldsymbol{T}} = T\vec{\boldsymbol{n}} = T\frac{\overrightarrow{\boldsymbol{AB}}}{\overline{AB}}$
- Roots of a complex number: $\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta+2\pi k}{n} + i\sin\frac{\theta+2\pi k}{n}\right)$ where; $k = 0, 1, 2, \cdots$