THE PAPUA NEW GUINEA UNIVERSITY OF TECHNOLOGY

DEPARTMENT OF MATHEMATICS & COMPUTER SCIENCE

**SECOND SEMESTER EXAMINATIONS – JUNE 2021**

**THIRD YEAR BACHELOR OF SCIENCE IN COMPUTER SCIENCE**

# CS313 – NUMERICAL METHODS

**TIME ALLOWED: 3 HOURS**

## INFORMATION FOR CANDIDATES

1. Write your name and student number clearly on the front of the examination answer booklet.
2. You have 10 minutes to read this paper. You must not begin writing during this time.
3. This paper contains FIVE (5) questions. **You should answer only Four (4) questions. Questions 1, 2 and 3 are mandatory**. You are to choose either Question 4 or Question 5 as your fourth question. You will be penalized if you answer both.
4. All answers must be written in examination answer booklets provided. No other written materials will be accepted.
5. Start the answer for each question on a new page. Do not use red ink.
6. Notes, textbooks, mobile phones and other recording devices are not allowed in the examination room.
7. Scientific and business calculators are allowed in the examination room.
8. Two pages of student information sheet are provided at the end of the examination paper.

## MARKING SCHEME

Marks are indicated at the beginning of each question. Total mark is 70.

**Question 1 [20 marks]**

Consider the following initial-value problem.

$y' = \frac{2-2xy}{1+x^2}, \quad 0 \le x \le 1, \quad y(0) = 1, \quad h = 0.5$

Use the Runge-Kutta method of order four to approximate the solutions IVP at $y(1.0)$, and compare the result to the actual value.

{ N.B. The actual value can be found using the solution $y(x) = \frac{2x+1}{x^2+1}$ }

**Question 2 [6 + 6 + 2 + 6 = 20 marks]**

Consider the following linear system,

$$\begin{cases} 4x_1 + x_2 + 2x_3 = 9 \\ 2x_1 + 4x_2 - x_3 = -5 \\ x_1 + x_2 - 3x_3 = -9 \end{cases}$$

(a) Find the first two iterations of the Jacobi method using $X^{(0)} = 0$,
(b) Find the first two iterations of the Gauss-Seidel method using $X^{(0)} = 0$,
(c) If the exact solutions were $x_1 = 1, \ x_2 = -1$ and $x_3 = 3$. Which of the two method about converges quicker towards the answer?
(d) If the linear system above represents a SOR system. Write out the three equations to solve $x_1$, $x_2$ and $x_3$ using the algorithm given on the formula sheet.

**Question 3 [10 + 5 = 15 marks]**

A natural cubic spline $S$ is defined by

$$S(x) = \begin{cases} S_0(x) = 1 + b_0(x-1) - d_0(x-1)^3 & x \in [1,2] \\ S_1(x) = 1 + b_1(x-2) - \frac{3}{4}(x-2)^2 + d_1(x-2)^3 & x \in [2,3] \end{cases}$$

(a) If $S$ interpolates the data $(1,1)$, $(2,1)$ and $(3,0)$ find $b_0$, $d_0$, $b_1$ and $d_1$.
(b) Explain what would be the difference if the boundary conditions were *clamped cubic spline*.

**Question 4 [15 marks]**

Approximate the following integral (for N=4), $\int_{-0.25}^{0.25} [\cos(x)]^2 \, dx$ using

(i) Mid-point rule,
(ii) Trapezoidal rule,
(iii) Simpsons rule.

**Question 5 [15 marks]**

The eigenvalues of matrix $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $b > 0, \quad c \geq 0$ are equal.

Prove that $a = d$ and $c = 0$.

Student Information Sheet

**Natural Cubic Splines for 3 points**

Suppose that $x_0 < x_1 < x_2$ and $f(x_0) = y_0$, $f(x_1) = y_1$ and $f(x_2) = y_2$. The natural cubic spline interpolation passing these points will be in the following form

$$S(x) = \begin{cases} S_0(x) = a_0 + b_0(x - x_0) + c_0(x - x_0)^2 + d_0(x - x_0)^3 & x \in [x_0, x_1] \\ S_1(x) = a_1 + b_1(x - x_1) + c_1(x - x_1)^2 + d_1(x - x_1)^3 & x \in [x_1, x_2] \end{cases}$$

Where the unknowns $a_0$, $b_0$, $c_0$, $d_0$ and $a_1$, $b_1$, $c_1$, $d_1$ can be found by solving the systems of equations that is produced by the following equations

$$\begin{cases} S_0(x_0) = y_0 \\ S_0(x_1) = y_1 \end{cases} \quad \begin{cases} S_1(x_1) = y_1 \\ S_1(x_2) = y_2 \end{cases} \quad \begin{cases} S_0'(x_1) = S_1'(x_1) \\ S_0''(x_1) = S_1''(x_1) \end{cases} \quad \begin{cases} S_0''(x_0) = 0 \\ S_1''(x_2) = 0 \end{cases}$$

Given the system of equation

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \quad \text{, where } |a_{11}| > |a_{12}|,\ |a_{11}| > |a_{13}|$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \quad |a_{22}| > |a_{21}|,\ |a_{22}| > |a_{23}|$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \quad |a_{33}| > |a_{31}|,\ |a_{33}| > |a_{32}|$$

Two iterative methods, the Jacobi method and the Gauss – Seidel Method can be used to solve the above system

**Jacobi method:**

$$x_1^{(k+1)} = \frac{b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)}}{a_{11}}$$

$$x_2^{(k+1)} = \frac{b_2 - a_{21}x_1^{(k)} - a_{23}x_3^{(k)}}{a_{22}}$$

$$x_3^{(k+1)} = \frac{b_3 - a_{31}x_1^{(k)} - a_{32}x_2^{(k)}}{a_{33}}$$

**Gauss – Seidel Method:**

$$x_1^{(k+1)} = \frac{b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)}}{a_{11}}$$

$$x_2^{(k+1)} = \frac{b_2 - a_{21}x_1^{(k+1)} - a_{23}x_3^{(k)}}{a_{22}}$$

$$x_3^{(k+1)} = \frac{b_3 - a_{31}x_1^{(k+1)} - a_{32}x_2^{(k+1)}}{a_{33}}$$

**Successive Over-Relaxation Method (SOR)**

$$x_i^{(k)} = (1 - \omega)x_i^{(k-1)} + \frac{\omega}{a_{ii}}\left[-\sum_{j=1}^{i-1}\left(a_{ij}x_j^{(k)}\right) - \sum_{j=i+1}^{n}\left(a_{ij}x_j^{(k-1)}\right) + b_i\right] \quad , \qquad for\ \omega > 1$$

**Runge-Kutta of Order four (4)**

To solve the initial value problem

$\frac{dy}{dx} = f(x, y) \qquad a \le x \le b \qquad y(a) = y_0$

Using the Runge-Kutta method we will use the following formulae

$K_i = hf(x_i, y_i)$

$U_i = hf(x_i + \frac{h}{2}, y_i + \frac{K_i}{2})$

$T_i = hf(x_i + \frac{h}{2}, y_i + \frac{U_i}{2})$

$A_i = hf(x_{i+1}, y_i + T_i)$

$y_{i+1} = y_i + \frac{1}{6}[K_i + 2U_i + 2T_i + A_i]$

**Mid-point Rule**

Assume that $f(x)$ is continuous on $[a, b]$. Let $n$ be a`positive integer and $h = \Delta x = \frac{b-a}{n}$. If $[a, b]$ is divided into $n$ subintervals, each length $h = \Delta x$, and $m_i$ is the midpoint of the i-th subinterval.

Set $M_n = \sum_{i=1}^{n} f(m_i)\Delta x_i$ , then $\lim_{n\to\infty} M_n = \int_a^b f(x)dx$.

**Trapezoidal Rule**

Assume that $f(x)$ is continuous on $[a, b]$. Let $n$ be a`positive integer and $h = \Delta x = \frac{b-a}{n}$. Let $[a, b]$ be divided into $n$ sub-intervals, each of length $h = \Delta x$, with end point at $P = \{x_0, x_1, x_2 \dots x_n\}$.

Set $T_n = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + \dots + 2f(x_{n-1}) + f(x_n)]$ , then $\lim_{n\to\infty} T_n = \int_a^b f(x)dx$.

**Simpson Rule**

Assume that $f(x)$ is continuous on $[a, b]$. Let $n$ be a`positive even integer and $h = \Delta x = \frac{b-a}{n}$. Let $[a, b]$ be divided into $n$ sub-intervals, each of length $h = \Delta x$, with end point at $P = \{x_0, x_1, x_2 \dots x_n\}$.

Set $S_n = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \dots + 2f(x_{n-2}) + 2f(x_{n-1}) + f(x_n)]$ , then $\lim_{n\to\infty} S_n = \int_a^b f(x)dx$.