THE PAPUA NEW GUINEA UNIVERSITY OF TECHNOLOGY

DEPARTMENT OF MATHEMATICS & COMPUTER SCIENCE

SECOND SEMESTER EXAMINATIONS - 2023

FIRST-YEAR ELECTRICAL AND MINING ENGINEERING AND MINERAL PROCESSING

EN121A – ENGINEERING MATHEMATICS II

TIME ALLOWED: 3 HOURS

INFORMATION FOR CANDIDATES

1. Write your name and student number clearly on the front of the examination booklet.
2. You have 10 minutes to read this paper. You must not begin writing during this time.
3. **Answer Question Five (Q5). Which is a compulsory question. Also choose and answer any other 3 questions. A total of four (4) questions must be answered including Q5. Show ALL working out.**
4. All answers must be written in examination booklets only. No other written material will be accepted.
5. Start the answer for each question on a **new** page. Do **not** use red ink.
6. Notes and textbooks are not allowed in the examination room. All mobile phones and electronic/recording devices must be switched off during the examination.
7. Scientific calculators are allowed in the examination room.
8. A three-page formula sheet is attached.

MARKING SCHEME

Marks are indicated at the beginning of each question. All questions carry equal marks.

Question 1 MATRICES (25 marks)

a) Given,

$$\begin{bmatrix} 0.5 & 0 & -0.5 \\ 1 & -0.2 & 0.3 \\ 0.5 & 0 & -1.5 \end{bmatrix}$$

i) Calculate the inverse from $\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \left[A_{jk}\right]^T$ where A_{jk} is the minor of a_{jk} in a $\det \mathbf{A}$ (the adjoint). **(10 marks)**

ii) Check by using $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$ (Show all steps). **(5 marks)**

b) Find the eigenvalues and eigenvectors of the following matrix. **10 marks)**

$$\begin{bmatrix} 10 & -4 \\ -12 & 18 \end{bmatrix}$$

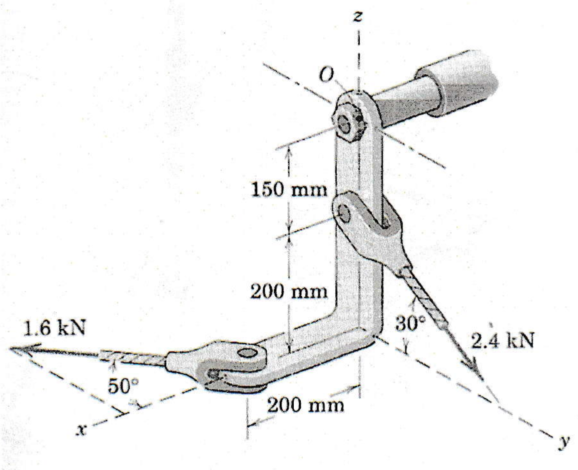
Question 2 VECTORS (25 marks)

a) In the context of a right-handed Cartesian coordinate system, given vectors $\mathbf{a} = [1, 2, 0]$ and $\mathbf{b} = [2, 3, 4]$. Find the following expressions.

i) $\mathbf{a} \times \mathbf{b}$, **(3 marks)**

ii) $\mathbf{a} \cdot \mathbf{b}$ **(3 marks)**

b)



Calculate the unit vectors, of the:

a) force **R** and **(9 marks)**

b) couple **M**, **(10 marks)**

exerted by the nut and bolt on the loaded bracket at *O*, to maintain equilibrium.

Question 3 BASIC LINEAR HOMOGENEOUS 1ST ORDER ODE (25 marks)

a) Differential equations $y' = f(ax + by + k)$ can be made separable by using as a new unknown function $v(x) = ax + by + k$. Using this method, solve $y' = (x + y - 2)^2$ **(12 marks)**

b) The population of a town grows at a rate proportional to the population present at time t. The initial population of 500 increases by 15% in 10 years.

i) What will be the population in 30 years? **(10 marks)**

ii) How fast is the population growing at $t = 30$ years? **(3 marks)**

Question 4 1ST AND 2ND ORDER ODE (25 marks)

a) The follow homogeneous equations can be classed as exact differential equations.

$$(\cot y + x^2)dx = x \csc^2 y \, dy.$$

As such,

i) demonstrate exactness, **(4 marks)**

ii) find the implicit solution u by appropriate integration and find the constant function, **(5 marks)**

iii) check your answer by implicit differentiation. **(4 marks)**

b) Find the transient motion (full solution) of the mass-spring system governed by the given equation. Show the details of your work.

$$y'' + 3y = 11 \cos 0.5t$$

(12 marks)

Question 5 THE LAPLACE TRANSFORM (25 marks)

a) A piecewise continuous function is given as follows:

$$f(t) = \begin{cases} 2 & \text{if } 0 < t < \pi \\ 0 & \text{if } \pi < t < 2\pi \\ \sin t & \text{if } t > 2\pi \end{cases}$$

i) Sketch $f(t)$, **(3 marks)**

ii) Write $f(t)$ in terms of the Heaviside or unit step function $u(t - a)$, **(5 marks)**

iii) Find the Laplace transform of the function. **(5 marks)**

b) In the following system of differential equations, solve the given initial value problem by means of Laplace transforms.

$y_1' + y_2 = 2 \cos t, \quad y_1 + y_2' = 0, \quad y_1(0) = 0, \quad y_2(0) = 1$ **(12 marks)**

DATA SHEET for EN121A EXAM 2023 SEMESTER 2

Derivatives and Integrals

1 $\frac{d}{dx}(x^n) = nx^{n-1}$ $\quad\therefore \int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad \left\{\begin{matrix}\text{provided}\\ n \neq -1\end{matrix}\right\}$

2 $\frac{d}{dx}(\ln x) = \frac{1}{x}$ $\quad\therefore \int \frac{1}{x}\,dx = \ln x + C$

3 $\frac{d}{dx}(e^x) = e^x$ $\quad\therefore \int e^x\,dx = e^x + C$

4 $\frac{d}{dx}(e^{kx}) = ke^{kx}$ $\quad\therefore \int e^{kx}\,dx = \frac{e^{kx}}{k} + C$

5 $\frac{d}{dx}(a^x) = a^x \ln a$ $\quad\therefore \int a^x\,dx = \frac{a^x}{\ln a} + C$

6 $\frac{d}{dx}(\cos x) = -\sin x$ $\quad\therefore \int \sin x\,dx = -\cos x + C$

7 $\frac{d}{dx}(\sin x) = \cos x$ $\quad\therefore \int \cos x\,dx = \sin x + C$

8 $\frac{d}{dx}(\tan x) = \sec^2 x$ $\quad\therefore \int \sec^2 x\,dx = \tan x + C$

9 $\frac{d}{dx}(\cosh x) = \sinh x$ $\quad\therefore \int \sinh x\,dx = \cosh x + C$

10 $\frac{d}{dx}(\sinh x) = \cosh x$ $\quad\therefore \int \cosh x\,dx = \sinh x + C$

11 $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ $\quad\therefore \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$

12 $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$ $\quad\therefore \int \frac{-1}{\sqrt{1-x^2}}\,dx = \cos^{-1} x + C$

13 $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$ $\quad\therefore \int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$

14 $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2+1}}$ $\quad\therefore \int \frac{1}{\sqrt{x^2+1}}\,dx = \sinh^{-1} x + C$

15 $\frac{d}{dx}(\cosh^{-1}) = \frac{1}{\sqrt{x^2-1}}$ $\quad\therefore \int \frac{1}{\sqrt{x^2-1}}\,dx = \cosh^{-1} x + C$

16 $\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$ $\quad\therefore \int \frac{1}{1-x^2}\,dx = \tanh^{-1} x + C$

More derivatives

$\frac{d}{dx}[\tan x] = \sec^2 x \qquad \frac{d}{dx}[\sec x] = \sec x \tan x$

$\frac{d}{dx}[\cot x] = -\csc^2 x \qquad \frac{d}{dx}[\csc x] = -\csc x \cot x$

Specific integrals

$$\int \tan x\,dx = \ln|\sec x| + C$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$

TRIGONOMETRIC SUBSTITUTIONS

EXPRESSION IN THE INTEGRAND	SUBSTITUTION	RESTRICTION ON θ	SIMPLIFICATION
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$-\pi/2 \le \theta \le \pi/2$	$a^2 - x^2 = a^2 - a^2\sin^2\theta = a^2\cos^2\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$-\pi/2 < \theta < \pi/2$	$a^2 + x^2 = a^2 + a^2\tan^2\theta = a^2\sec^2\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$\begin{cases} 0 \le \theta < \pi/2 & (\text{if } x \ge a) \\ \pi/2 < \theta \le \pi & (\text{if } x \le -a) \end{cases}$	$x^2 - a^2 = a^2\sec^2\theta - a^2 = a^2\tan^2\theta$

Reduction formula

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx$$

$$\int \tan^n x\,dx = \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x\,dx$$

$$\int \sec^n x\,dx = \frac{\sec^{n-2}x\tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx$$

The guide to completing the square is $x^2 + bx = \left[x + \left(\frac{b}{2}\right)^2\right] - \left(\frac{b}{2}\right)^2$

INTEGRATING PRODUCTS OF TANGENTS AND SECANTS

$\int \tan^m x \sec^n x\, dx$	PROCEDURE	RELEVANT IDENTITIES
n even	• Split off a factor of $\sec^2 x$. • Apply the relevant identity. • Make the substitution $u = \tan x$.	$\sec^2 x = \tan^2 x + 1$
m odd	• Split off a factor of $\sec x \tan x$. • Apply the relevant identity. • Make the substitution $u = \sec x$.	$\tan^2 x = \sec^2 x - 1$
$\begin{cases} m \text{ even} \\ n \text{ odd} \end{cases}$	• Use the relevant identities to reduce the integrand to powers of $\sec x$ alone. • Then use the reduction formula for powers of $\sec x$.	$\tan^2 x = \sec^2 x - 1$

Trigonometrical identities

(a) $\sin^2\theta + \cos^2\theta = 1$; $\sec^2\theta = 1 + \tan^2\theta$; $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$

(b) $\sin(A+B) = \sin A\cos B + \cos A\sin B$

$\sin(A-B) = \sin A\cos B - \cos A\sin B$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\cos(A-B) = \cos A\cos B + \sin A\sin B$

$\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

$\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$

(c) Let $A = B = \theta$ $\therefore$ $\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$

$\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$

Hyperbolic identities

$\cosh x + \sinh x = e^x$

$\cosh x - \sinh x = e^{-x}$

$\cosh^2 x - \sinh^2 x = 1$

$1 - \tanh^2 x = \operatorname{sech}^2 x$

$\coth^2 x - 1 = \operatorname{csch}^2 x$

$\cosh(-x) = \cosh x$

$\sinh(-x) = -\sinh x$

$\sinh(x+y) = \sinh x\cosh y + \cosh x\sinh y$

$\cosh(x+y) = \cosh x\cosh y + \sinh x\sinh y$

$\sinh(x-y) = \sinh x\cosh y - \cosh x\sinh y$

$\cosh(x-y) = \cosh x\cosh y - \sinh x\sinh y$

$\sinh 2x = 2\sinh x\cosh x$

$\cosh 2x = \cosh^2 x + \sinh^2 x$

$\cosh 2x = 2\sinh^2 x + 1 = 2\cosh^2 x - 1$

- **Kirchhoff's current law: The algebraic sum of all currents entering and exiting a node must equal zero.**
- **Kirchhoff's voltage law: the voltage around a loop equals the sum of every voltage drop in the same loop for any closed network and equals zero. Ohm's law is: $V = IR$.**
- $\boldsymbol{i\cdot i = j\cdot j = k\cdot k = 1}$ & $\boldsymbol{i\cdot j = j\cdot i = i\cdot k = k\cdot i = j\cdot k = k\cdot j = 0}$
- $\boldsymbol{i\times j = k, j\times k = i, k\times i = j}$ & $\boldsymbol{j\times i = -k, k\times j = -i, i\times k = -j}$ &
  $\boldsymbol{i\times i = j\times j = k\times k = 0}$
- In vector algebra: $\vec{T} = T\vec{n} = T\dfrac{\overrightarrow{AB}}{\overline{AB}}$

Steps for solving the exact differential equation $u(x,y) = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$

1. Test for exactness: $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$
2. Find implicit solution $u = \int M dx + k(y)$ or $u = \int N dy + l(x)$. Then find $k'(y)$ or $l'(x)$ with respect to its independent variable alone, and equate to M or N. Then integrate to find value of k or l.
3. Check by implicit differentiation to see if you get the original differential equation $u = Mdx + Ndy$.

For a non-exact 1st order ODE: $Pdx + Qdy = 0$, multiply throughout by integrating factor F to get exact ODE, $FPdx + FQdy = 0$

F can be calculated as $e^{\int R(x)dx}$ where $R = \frac{1}{Q}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$

For the 1st order non-homogeneous equation, $y' + py = r$,
y is given as $e^{-h}[\int e^h r dx + c]$ where $h = \int p dx$

For the 1st order non-linear Bernoulli equation $y' + py = gy^a$ set $u = y^{1-a}$ differentiate this and substitute for y' and y^{1-a} to get as linear nonhomogeneous equation.

Stroud's guide for solving linear 2nd order ode:

1 Solution of equations of the form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$

(1) Auxiliary equation: $am^2+bm+c=0$

(2) Types of solutions:

(a) Real and different roots $\quad m=m_1$ and $m=m_2$

$y=Ae^{m_1x}+Be^{m_2x}$

(b) Real and equal roots $\quad m=m_1$ (twice)

$y=e^{m_1x}(A+Bx)$

(c) Complex roots $\quad m=\alpha\pm j\beta$

$y=e^{\alpha x}(A\cos\beta x+B\sin\beta x)$

2 Equations of the form $\frac{d^2y}{dx^2}+n^2y=0$

$y=A\cos nx+B\sin nx$

3 Equations of the form $\frac{d^2y}{dx^2}-n^2y=0$

$y=A\cosh nx+B\sinh nx$

4 General solution

$y=$ complementary function $+$ particular integral

Kreyszig's guide to solving y_p

Table 2.1 **Method of Undetermined Coefficients**

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n\ (n=0,1,\cdots)$	$K_nx^n+K_{n-1}x^{n-1}+\cdots+K_1x+K_0$
$k\cos\omega x$ $k\sin\omega x$	$K\cos\omega x+M\sin\omega x$
$ke^{\alpha x}\cos\omega x$ $ke^{\alpha x}\sin\omega x$	$e^{\alpha x}(K\cos\omega x+M\sin\omega x)$

Stroud's guide to solving y_p

If	Assume
$f(x)=k\ldots\ \ldots$	$y=C$
$f(x)=kx\ldots\ \ldots$	$y=Cx+D$
$f(x)=kx^2\ldots\ \ldots$	$y=Cx^2+Dx+E$
$f(x)=k\sin x$ or $k\cos x$	$y=C\cos x+D\sin x$
$f(x)=k\sinh x$ or $k\cosh x$	$y=C\cosh x+D\sinh x$
$f(x)=e^{kx}\ldots\ \ldots$	$y=Ce^{kx}$

Brief but required table of Laplace transforms:

	$f(t)$	$\mathcal{L}(f)$		$f(t)$	$\mathcal{L}(f)$
1	1	$1/s$	7	$\cos\omega t$	$\frac{s}{s^2+\omega^2}$
2	t	$1/s^2$	8	$\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$
3	t^2	$2!/s^3$	9	$\cosh at$	$\frac{s}{s^2-a^2}$
4	t^n $(n=0,1,\cdots)$	$\frac{n!}{s^{n+1}}$	10	$\sinh at$	$\frac{a}{s^2-a^2}$
5	t^a (a positive)	$\frac{\Gamma(a+1)}{s^{a+1}}$	11	$e^{at}\cos\omega t$	$\frac{s-a}{(s-a)^2+\omega^2}$
6	e^{at}	$\frac{1}{s-a}$	12	$e^{at}\sin\omega t$	$\frac{\omega}{(s-a)^2+\omega^2}$

The Second Shifting Theorem: $\mathcal{L}(\mathcal{U}(t-a)g(t))=e^{-sa}\mathcal{L}(g(t+a))$

Generally any Laplace transform of a derivative can be written as:

$$\mathcal{L}(f^{(n)})=s^n\mathcal{L}(f)-s^{n-1}f(0)-s^{n-2}f'(0)-\cdots-f^{(n-1)}(0)$$